Eigenvalues 3

Given an automorphism in $\mathbb{P}_2$ the vector space of 2nd degree polynoms named f.
1. Obtain its kernel and image.
2. Knowing that x+1 transforms into itself, that 3 transforms into -3 and that the determinant of the matrix is 6, obtain the eigenvalues of the automorphism.
3. knowing that $L\{(x^2+x+1)\}$ is an eigenspace, obtain F_B.

Note: You cannot use information of later points of the problem in previous points.

1. $Ker(f)=\{\bar{0}\}$ $\quad \dim(Ker(f))=0$

$Im(f)=\mathbb{P}_2$ $\quad \dim(Im(f))=3$

2. $S(-1)=\mathcal{L}\{1\}$ $\quad MO(-1)=1$

$S(1)=\mathcal{L}\{x+1\}$ $\quad MO(1)=1$

$S(-6)=\mathcal{L}\{x^2+x+1\}$ $\quad MO(-6)=1$

from 3.

3. F_B?

$$F_B=\begin{array}{ccc} \begin{pmatrix} -6 & 0 & 0 \\ -7 & 1 & 0 \\ -7 & 2 & -1 \end{pmatrix} \\ f(x^2) \quad f(x) \quad f(1) \end{array}$$

in $B=\{x^2, x, 1\}$

$f(1)=-1$

$f(x+1)=x+1 \longrightarrow f(x)+\underbrace{f(1)}_{-1}=x+1 \rightarrow f(x)=x+2$

$f(x^2+x+1)=-6x^2-6x-6 \longrightarrow f(x^2)+\underbrace{f(x)}_{x+2}+\underbrace{f(1)}_{-1}=-6x^2-6x-6$

$f(x^2)=-6x^2-7x-7$

Eigenvalues 4 Given an endomorphism in $\mathbb{R}^3$ where $f(\bar{x}) = (x^1+x^2,\ x^1+x^3,\ x^2+x^3)$.

1. Prove it's an Endomorphism.
2. Obtain $Ker(f)$ and $Im(f)$.
3. Obtain $S(\lambda_i)\ \forall i$
4. Is f diagonalizable?

$f: \mathbb{R}^3 \longrightarrow \mathbb{R}^3$

$\forall \alpha, \beta \in \mathbb{R} \wedge \forall \bar{x}, \bar{y} \in \mathbb{R}^3$

f is a homomorphism $\overset{?}{\Longleftrightarrow}$ $\boxed{f(\alpha\bar{x}+\beta\bar{y})} \overset{?}{=} \boxed{\alpha f(\bar{x}) + \beta f(\bar{y})}$

$$\boxed{f(x^1, x^2, x^3) = (x^1+x^2,\ x^1+x^3,\ x^2+x^3)}$$

$$\boxed{f(\alpha\bar{x}+\beta\bar{y})} = f(\alpha(x^1,x^2,x^3)+\beta(y^1,y^2,y^3)) = f\left[(\alpha x^1+\beta y^1,\ \alpha x^2+\beta y^2,\ \alpha x^3+\beta y^3)\right] =$$

$\bar{x} = (x^1, x^2, x^3)_B$

$\bar{y} = (y^1, y^2, y^3)_B$

$B = \{\bar{e}_1, \bar{e}_2, \bar{e}_3\} \subset \mathbb{R}^3$

$$= \left(\alpha x^1+\beta y^1+\alpha x^2+\beta y^2,\ \alpha x^1+\beta y^1+\alpha x^3+\beta y^3,\ \alpha x^2+\beta y^2+\alpha x^3+\beta y^3\right) =$$

$$= \left(\alpha(x^1+x^2)+\beta(y^1+y^2),\ \alpha(x^1+x^3)+\beta(y^1+y^3),\ \alpha(x^2+x^3)+\beta(y^2+y^3)\right) =$$

$$= \left(\alpha(x^1+x^2),\ \alpha(x^1+x^3),\ \alpha(x^2+x^3)\right) + \left(\beta(y^1+y^2),\ \beta(y^1+y^3),\ \beta(y^2+y^3)\right) =$$

$$= \alpha(x^1+x^2,\ x^1+x^3,\ x^2+x^3) + \beta(y^1+y^2,\ y^1+y^3,\ y^2+y^3) = \boxed{\alpha f(\bar{x}) + \beta f(\bar{y})}$$

2.

$$F_B = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

$f(\bar{e}_1)\ f(\bar{e}_2)\ f(\bar{e}_3)$ in $\mathcal{B}$

$$\mathcal{B}: \begin{cases} \bar{e}_1 = (1,0,0)_B \\ \bar{e}_2 = (0,1,0)_B \\ \bar{e}_3 = (0,0,1)_B \end{cases}$$

$$\begin{vmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{vmatrix} = -1-1 = -2 \neq 0$$

$\dim(\mathrm{Im}(f)) = \mathrm{Rg}(F_B) = 3 \longrightarrow \dim(\mathrm{Im}(f)) = \dim(\mathbb{R}^3) \rightarrow \mathrm{Im}(f) = \mathbb{R}^3$

f is SUPRAYECTIVE

$$\underbrace{\dim(\mathrm{Im}(f))}_{3} + \dim(\mathrm{Ker}(f)) = \underbrace{\dim(\mathbb{R}^3)}_{3}$$

$\dim(\mathrm{Ker}(f)) = 3-3 = 0 \longrightarrow \mathrm{Ker}(f) = \{\bar{0}\}$

f is INYECTIVE

$f: V_1 \to V_2$	$V_1 \neq V_2$	$V_1 = V_2$
Not Biyective	Homomorph.	Endomorph.
Biyective	Isomorph.	Automorph.

f is INYECTIVE + f is SUPRAYECTIVE $\longrightarrow$ f is BIYECTIVE $\longrightarrow$ f is an AUTOMORPHISM

3.

$$|F_B - \lambda I| = 0 \longrightarrow \begin{vmatrix} 1-\lambda & 1 & 0 \\ 1 & -\lambda & 1 \\ 0 & 1 & 1-\lambda \end{vmatrix} = 0 \longrightarrow (1-\lambda)\left[(\lambda^2-\lambda)-1\right] - (1-\lambda) = 0$$

$$\lambda^2 - \lambda - 1 - \lambda^3 + \lambda^2 + \lambda - 1 + \lambda = 0 \longrightarrow -\lambda^3 + 2\lambda^2 + \lambda - 2 = 0$$

$$\begin{array}{r|rrrr} & -1 & 2 & 1 & -2 \\ 1 & & -1 & 1 & 2 \\ \hline & -1 & 1 & 2 & \boxed{0} \end{array}$$

$$\begin{array}{r|rrr} & -1 & 1 & 2 \\ -1 & & 1 & -2 \\ \hline & -1 & 2 & \boxed{0} \end{array}$$

$$(\lambda-1)(\lambda+1)(\lambda-2) = 0$$

$$F_B = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

$f(\bar{e}_1)$ $f(\bar{e}_2)$ $f(\bar{e}_3)$

$\lambda_1 = -1 \longrightarrow MO(-1) = 1$

$\lambda_2 = 1 \longrightarrow MO(1) = 1$

$\lambda_3 = 2 \longrightarrow MO(2) = 1$

$$\boxed{S(\lambda) = Ker(f - \lambda i)}$$

$f(\bar{e}_1) + f(\bar{e}_2) + f(\bar{e}_3) = (2,2,2)_B$

$f(\bar{e}_1 + \bar{e}_2 + \bar{e}_3) = (2,2,2)_B$

$f((1,1,1)) = (2,2,2) = 2(1,1,1)$

$1 \leq \dim(S(-1)) \leq \cancel{MO(-1)}\ 1 \longrightarrow \dim(S(-1)) = 1$

$1 \leq \dim(S(1)) \leq \cancel{MO(1)}\ 1 \longrightarrow \dim(S(1)) = 1$

$1 \leq \dim(S(2)) \leq \cancel{MO(2)}\ 1 \longrightarrow \dim(S(2)) = 1$

$S(-1) = Ker(f + i)$

$$\underbrace{\begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix}}_{F_B + I} \underbrace{\begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix}}_{\bar{x}} = \underbrace{\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}}_{\bar{0}}$$

$2x^1 + x^2 = 0$

$\cancel{x^1 + x^2 + x^3 = 0}$

$x^2 + 2x^3 = 0$

$$\begin{cases} x^1 = -\frac{1}{2}\alpha \\ x^2 = \alpha \\ x^3 = -\frac{1}{2}\alpha \end{cases}$$

$\forall \alpha \in \mathbb{R}$

$S(-1) = \mathcal{L}\{\underbrace{(1,-2,1)}_{\bar{u}_1}\}$

$S(1) = Ker(f - i)$

$$\underbrace{\begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix}}_{F_B - I} \underbrace{\begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix}}_{\bar{x}} = \underbrace{\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}}_{\bar{0}}$$

$x^2 = 0$

$x^1 - x^2 + x^3 = 0$

$\cancel{x^2 = 0}$

$$\begin{cases} x^1 = \beta \\ x^2 = 0 \\ x^3 = -\beta \end{cases}$$

$\forall \beta \in \mathbb{R}$

$S(1) = \mathcal{L}\{\underbrace{(1,0,-1)}_{\bar{u}_2}\}$

$S(2) = \mathrm{Ker}(f - 2i)$

$$\underbrace{\begin{pmatrix} -1 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{pmatrix}}_{F_B - 2I} \underbrace{\begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix}}_{\bar{X}} = \underbrace{\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}}_{\bar{0}}$$

$$\begin{aligned} -x^1 + x^2 &= 0 \\ \cancel{x^1 - 2x^2 + x^3} &\cancel{= 0} \\ x^2 - x^3 &= 0 \end{aligned}$$

$$\begin{cases} x^1 = \gamma \\ x^2 = \gamma \\ x^3 = \gamma \end{cases} \quad \forall \gamma \in \mathbb{R}$$

$$S(2) = \mathcal{L}\left\{\underbrace{(1,1,1)}_{\bar{u}_3}\right\}$$

4.

$$S(-1) = \mathcal{L}\left\{\underbrace{(1,-2,1)}_{\bar{u}_1}\right\}$$

$$S(1) = \mathcal{L}\left\{\underbrace{(1,0,-1)}_{\bar{u}_2}\right\}$$

$$S(2) = \mathcal{L}\left\{\underbrace{(1,1,1)}_{\bar{u}_3}\right\}$$

Since $\bar{u}_1, \bar{u}_2$ and $\bar{u}_3$ are L.I.

Since $\bar{u}_1, \bar{u}_2$ and $\bar{u}_3$ are Eigenvectors

$$B' = \left\{\underbrace{\bar{u}_1}_{\in S(-1)}, \overbrace{\bar{u}_2}^{\in S(1)}, \underbrace{\bar{u}_3}_{\in S(2)}\right\}$$ Is a base of Eigenvectors of f in $\mathbb{R}^3$

$$B' = \begin{cases} \bar{u}_1 = (1,-2,1)_B = (1,0,0)_{B'} \\ \bar{u}_2 = (1,0,-1)_B = (0,1,0)_{B'} \\ \bar{u}_3 = (1,1,1)_B = (0,0,1)_{B'} \end{cases}$$

$$B = \begin{cases} \bar{e}_1 = (1,0,0)_B = (1/6, 1/2, 1/3)_{B'} \\ \bar{e}_2 = (0,1,0)_B = (-1/3, 0, 1/3)_{B'} \\ \bar{e}_3 = (0,0,1)_B = (1/6, -1/2, 1/3)_{B'} \end{cases}$$

$$\begin{tikzcd} B \arrow[r, bend left, "C"] & B' \arrow[l, bend left, "C^{-1}"] \end{tikzcd}$$

$$C = \begin{array}{c} \begin{pmatrix} 1 & 1 & 1 \\ -2 & 0 & 1 \\ 1 & -1 & 1 \end{pmatrix} \\ \bar{u}_1 \ \ \bar{u}_2 \ \ \bar{u}_3 \\ \text{in } B \end{array}$$

$$C^{-1} = \frac{\mathrm{Adj}(C)^t}{|C|}$$

$$C^{-1} = \begin{array}{c} \begin{pmatrix} 1/6 & -1/3 & 1/6 \\ 1/2 & 0 & -1/2 \\ 1/3 & 1/3 & 1/3 \end{pmatrix} \\ \bar{e}_1 \ \ \bar{e}_2 \ \ \bar{e}_3 \\ \text{in } B' \end{array}$$

Is f diagonalizable? Yes

$$f \text{ diagonalizable} \iff \exists \text{ Base of Eigenvectors} \iff \dim(S(\lambda_i)) = MO(\lambda_i) \quad \forall i$$

$$\boxed{B' = \{\underbrace{\bar{u}_1}_{\in S(-1)}, \overbrace{\bar{u}_2}^{\in S(1)}, \underbrace{\bar{u}_3}_{\in S(2)}\}}$$

$$\boxed{\begin{array}{l} 1 \leq \dim(S(-1)) \leq \cancel{MO(-1)}^{1} \longrightarrow \dim(S(-1)) = 1 \\ 1 \leq \dim(S(1)) \leq \cancel{MO(1)}^{1} \longrightarrow \dim(S(1)) = 1 \\ 1 \leq \dim(S(2)) \leq \cancel{MO(2)}^{1} \longrightarrow \dim(S(2)) = 1 \end{array}}$$

$$F_{B'} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

$f(\bar{u}_1)$ $f(\bar{u}_2)$ $f(\bar{u}_3)$ in B'

$f(\bar{u}_1) = -\bar{u}_1 \longrightarrow \bar{u}_1 \in S(-1) \quad -\bar{u}_1 = (-1, 0, 0)_{B'}$

$f(\bar{u}_2) = \bar{u}_2 \longrightarrow \bar{u}_2 \in S(1) \quad \bar{u}_2 = (0, 1, 0)_{B'}$

$f(\bar{u}_3) = 2\bar{u}_3 \longrightarrow \bar{u}_3 \in S(2) \quad 2\bar{u}_3 = (0, 0, 2)_{B'}$

$$\boxed{F_{B'} = C^{-1} F_B C}$$